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REALIZATION OF HOLONOMY IN THE MOTION OF MECHANISMS*

M.O. RAKHULA

A relationship between the modern theory of connectedness and the mechanics of non-holonomic systems is established by two examples.

The idea of connectedness is the following. A differential equation $y' = f(x, y)$ is considered as a field directions in the xy plane, while the system of differential equations

$$\partial u^\alpha/\partial u^i = -\Gamma_i^\alpha(u),\ i = 1, 2, \ldots, n;\ \alpha = n+1, \ldots, n+r$$

with right sides dependent on both the independent variables u^i (basis coordinates) and on the unknown functions u^α (laminary coordinates) in the general case, or equivalently, the Pfaff system $du^\alpha + \Gamma_i^\alpha du^i = 0$, is considered as a field of n-dimensional areas in the space of all variables (u^i, u^α). The connectedness in the stratification $\pi: V_1 \to V$ is the assignment of a horizontal distribution Δ_h in the manifold V_1, i.e., that field of areas which the tangential stratification TV /1/ covers in the tangential mapping $T\pi$. Critical for connectedness is the intersection of layers along paths given on the base V, and particularly the transformation of layers in the traversal of closed cylces, i.e., that which is called holonomy.

Two problems from the theory of mechanisms are considered below, which could be models to describe motions with non-holonomic constraints and reveal the value of holonomy in practical problems. The first problem illustrates the occurrence of connectedness and holonomy in an ordinary stratification, while the second demonstrates these concepts in a more complex double stratification. The multiple stratifications are not here assumed to be vectorial /2/. Such generality turns out to be more productive.

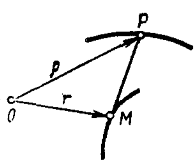

Fig.1

Problem 1. A certain curve $\boldsymbol{\rho}(t)$ is given on a plane (Fig.1). It is required to find the curve $\mathbf{r}(t)$ in this same plane for which the length of a tangent segment from the point of tangency M to the point of intersection P with the curve $\boldsymbol{\rho}(t)$ is a constant, and describes the trajectory of the point M as a function of the shape of the curve $\boldsymbol{\rho}(t)$.

Let (u, v) and (x, y) be the coordinates of the points P and M. The rod PM is identified with the point (u, v, x, y) of the four-space $\mathbf{R}^4$. The mapping $\pi: PM \to P$ projects $\mathbf{R}^4$ onto $\mathbf{R}^2$ stratifying $\mathbf{R}^4$ into two-dimensional layers. The coordinates (u, v) become basis coordinated, and (x, y) become laminary. The Point P belongs to the base, and the point M to the layer $\pi^{-1}(P)$. The conditions

1) $d\mathbf{r} \parallel \mathbf{r} - \rho$, 2) $|\mathbf{r} - \boldsymbol{\rho}| = a$ — const are equivalent to the Pfaff system

$$a^2 dx = (x-u)\,\theta,\ a^2 dy = (y-v)\,\theta\ \ (\theta = (x-u)\,du + (y-v)\,dv)$$

governing the two-dimensional distribution Δ_h in $\mathbf{R}^4$. The vector fields

$$X_1 = \frac{\partial}{\partial u} + \frac{x-u}{a^2} Z,\quad X_2 = \frac{\partial}{\partial v} + \frac{y-v}{a^2} Z$$
$$\left(Z = (x-u)\frac{\partial}{\partial x} + (y-v)\frac{\partial}{\partial y}\right)$$

comprise the basis for Δ_h. Together with the operators $X_3 = \partial\,\partial x$, $X_4 = \partial/\partial y$, the fields X_1 and X_2 form an adapted basis (/1/, p.169). The distribituion Δ_h is not integrable, i.e.,

$$[X_1 X_2] = -a^{-2}((y-v)\,X_3 - (x-u)\,X_4)$$

The connections characterizing the motion of the rod PM are non-holonomic (/3/, p.46). The vector field $[X_1 X_2]$ is an operator of the holonomy group; its trajectories are concentric circles in each layer $\pi^{-1}(P)$. A new layer over P can be extracted and taken by fixing a in one of these circles in $\pi^{-1}(P)$. Performing such a fixing in each layer, we obtain a stratification of the holonomy /4/ $\sigma = \pi \circ \varphi: \mathbf{R}^3 \to \mathbf{R}^2$ with the reducing mapping

$$\varphi: (u, v, \tau) \to (u, v, x, y),\ x = u + a\cos\tau,\ y = v + a\sin\tau$$

In the stratification of the holonomy the two Pfaff equations are replaced by one

$$a d\tau = \sin\tau du - \cos\tau dv.$$

In place of the distribution Δ_h there is the distribution Δ_h', also two-dimensional, with the

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vector basis

$$Y_1 = \frac{\partial}{\partial u} + \frac{1}{a} \sin \tau \frac{\partial}{\partial \tau}, \quad Y_2 = \frac{\partial}{\partial v} - \frac{1}{a} \cos \tau \frac{\partial}{\partial \tau}$$

The fields Y_1 and Y_2 are φ-connected with the fields X_1 and X_2: $T\varphi Y_k = X_k$, $k = 1,2$. The bracket $[Y_1Y_2] = a^{-2}\partial/\partial\tau$ is φ-connected to the bracket $[X_1X_2]$. Therefore, the point P in the stratification of the holonomy is a point of the base, its trajectory on the plane uv is a path on the base, the layer $\sigma^{-1}(P)$ is a circle of radius a with centre P, the layer above the path is a one-parameter family of circles, and the lifts are orthogonal trajectories of this family (Fig.2). If the path on the base is a straight line, then the lifts are ordinary tractrices.

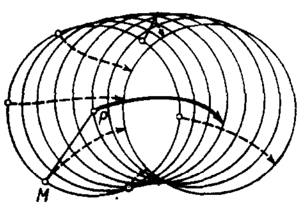

Fig.2

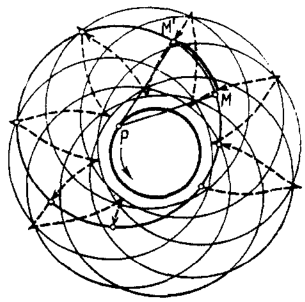

Fig.3

The dashed lines in Fig.3 display the lifts of paths having the form of a circle. The initial position of the rod PM and its new location PM' on completion of counter-clockwise traversal of the circle by the point P are shown. Rotation of the circle $M \to M'$ is indeed a manifestation of holonomy. It is clear that such a rotation of the circle $\sigma^{-1}(P)$ is observed when the point P traverses any closed cycle. This fact underlines the construction of the planimeter /5/.

Among such problems is a large number of problems of control theory, particularly pursuit games when the motion of one point $(x, y, \ldots)$ depends on the behaviour of another point $(u, v, \ldots)$ and the nature of the connection is described by a Pfaff system.

It should be noted that holonomy is always observed in the appearance of vector field brackets. For instance, if there are vector fields $X = x^i\partial/\partial u^i$ and $Y = y^i\partial/\partial u^i$, then R^n can be supplemented by two times axes t and s and the vector fields $X' = \partial/\partial t + X$ and $Y' = \partial/\partial s + Y$ in the stratification $R^{n+2} \to R^n$, $(u^i, t, s) \to (u^i)$ being formed, will form a connectedness with the bracket $[X'Y'] = [XY]$ as holonomy group operator.

In particular, the gyroscope effect, i.e. two rotations in R^3 around the x and y axes governed by the vector fields $X = z\partial/\partial y - y\partial/\partial z$ and $Y = z\partial/\partial x - x\partial/\partial z$ generate a third rotation around the z axis by interacting $[XY] = -y\partial/\partial x + x\partial/\partial y$, is a manifestation of holonomy.

We intestigate how holonomy appears in the double stratification situation defined by the commutative diagram displayed in Fig.4 where the arrows π_1, π_2, ρ_1, ρ_2 are assumed submersions.

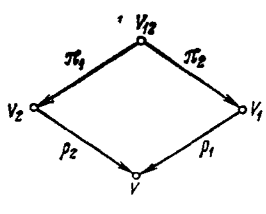

Fig.4

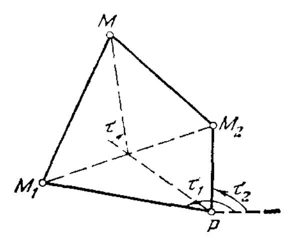

Fig.5

Problem 2. There is a rhombus PM_1M_2M in space whose sides are hinge-connected at the vertices (Fig.5). The points P, M_1 and M_2 move in a certain plane, where the points M_1 and M_2 follow the point P according to the condition in Problem 1. Describe the motion of the point M when it is known that the tangent to its trajectory lies in the M_1M_2M plane.

We first introduce the coordinates of the vertices: $P\ (u, v)$, $M_1\ (x_1, y_1)$, $M_2\ (x_2, y_2)$, $M\ (x, y, z)$ and we define the projections

$$PM_1M_2M \xrightarrow{\pi_1} PM_2 \xrightarrow{\rho_2} P, \quad PM_1M_2M \xrightarrow{\pi_2} PM_1 \xrightarrow{\rho_1} P.$$

The manifold V_{12} is nine-dimensional, the manifolds V_1 and V_2 are four-dimensional, and the base V is two-dimensional. By analogy with Problem 1, the mapping

$$\varphi : (u, v, \tau_1, \tau_2, \tau) \to (u, v, x_1, y_1, x_2, y_2, x, y, z)$$

where

$$x_k = u + a\cos\tau_k,\quad y_k = v + a\sin\tau_k;\quad k = 1,\ 2$$

$$x = u + a(\cos\tau_1 + \cos\tau_2)\cos^2\frac{\tau}{2}$$

$$y = v + a(\sin\tau_1 + \sin\tau_2)\cos^2\frac{\tau}{2}$$

$$z = a\cos\frac{\tau_1 - \tau_2}{2}\sin\tau$$

reduces the nine-space to five-dimensional, and reduces the conditions

$$1)\ dr_1 \| r_1 - \rho,\quad 2)\ dr_2 \| r_2 - \rho,\quad 3)\ (dr, r_1 - \rho, r_2 - \rho) = 0$$
$$4)\ |r_1 - \rho| = |r_2 - \rho| = |r - r_1| = |r - r_2| = a - \text{const}$$

to the Pfaff system

$$a\,d\tau_k = \sin\tau_k du - \cos\tau_k dv;\quad k = 1,\ 2$$
$$a\,d\tau = \sin\tau\cos\frac{\tau_1 - \tau_2}{2}\left(\cos\frac{\tau_1 + \tau_2}{2}du + \sin\frac{\tau_1 + \tau_2}{2}dv\right)$$

The latter is not fully integrable, but allows of the first integral

$$\sin\frac{\tau_1 - \tau_2}{2} = C\,\mathrm{tg}\,\frac{\tau}{2} \qquad (1)$$

(C is an arbitrary constant). This is explained by the fact that while the distribution Δ, exteneded over the vector fields

$$Y_1 = \frac{\partial}{\partial u} + \frac{1}{a}\left(\sin\tau_1\frac{\partial}{\partial\tau_1} + \sin\tau_2\frac{\partial}{\partial\tau_2}\right) + \frac{1}{2a}(\cos\tau_1 + \cos\tau_2)\sin\tau\frac{\partial}{\partial\tau}$$
$$Y_2 = \frac{\partial}{\partial v} - \frac{1}{a}\left(\cos\tau_1\frac{\partial}{\partial\tau_1} + \cos\tau_2\frac{\partial}{\partial\tau_2}\right) + \frac{1}{2a}(\sin\tau_1 + \sin\tau_2)\sin\tau\frac{\partial}{\partial\tau}$$

with the bracket $[Y_1Y_2] = a^{-2}(\partial/\partial\tau_1 + \partial/\partial\tau_2)$, is linearly independent of Y_1 and Y_2 and not integrable, the projections $Y_1' = Y_1 - \partial/\partial u$ and $Y_2' = Y_2 - \partial/\partial v$ on the three-dimensional layer $\sigma^{-1}(P)$, $\sigma = \varphi\rho_1\pi_2 = \varphi\rho_2\pi_1$, form a linearly-dependent system with the bracket $[Y_1'Y_2'] = [Y_1Y_2]$

$$\sin\frac{\tau_1 + \tau_2}{2}Y_1' + \cos\frac{\tau_1 + \tau_2}{2}Y_2' - a\cos\frac{\tau_1 - \tau_2}{2}[Y_1'Y_2'] = 0$$

To describe the motion of the point M we use the substitution $X = x - u,\ Y = y - v,\ Z = z$ to reduce (1) to the equation

$$(X^2 + Y^2 + Z^2)^2 = 4a^2(X^2 + Y^2 - C^2Z^2)$$

or

$$(\bar{X}^2 + \bar{Y}^2)^2 = 4a^2(\bar{X}^2 - C^2\bar{Y}^2),\quad (\bar{X}^2 = X^2 + Y^2,\quad \bar{Y} = Z)$$

The last equation defines a family of two-sheeted curves among which is lemniscate for $C = 1$. Rotating these curves around the $\bar{Y}$ axis, we obtain a family of surfaces of revolution with centre at the point P on which the three-dimensional layer $\sigma^{-1}(P)$ is stratified. Depending on the initial values of the angles τ_1, τ_2 and τ the point M slides over one of these surfaces (Fig.6). Its trajectory on such a surface depends on the path described by the point P. When the point P traverses a closed cycles, the point M returns to the initial parallel but occupies a new position M'. Under the influence of holonomy each parallel is rotated through the angle $M \to M'$.

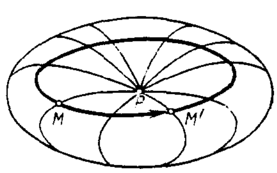

Fig.6

As is known /6/, connectedness can be defined by a ν-tuple stratification in the general case. For a binary stratification it is determined by giving two distributions Δ_h^1 and Δ_h^2 which are horizontal for the stratifications π_1 and π_2, respectively, on the total manifold V_{12}. In other words, if a vertical distribution

$$\Delta_v^1 = \mathrm{Ker}\,T\pi_1,\ \Delta_v^2 = \mathrm{Ker}\,T\pi_2,$$

is also considered on V_{12}, then this is equivalent to determining a structure

$$\Delta \oplus \Delta_1 \oplus \Delta_2 \oplus \Delta_{12}$$
$$(\Delta = \Delta_h^1 \cap \Delta_h^2,\ \Delta_1 = \Delta_h^2 \cap \Delta_v^1,\ \Delta_2 = \Delta_h^1 \cap \Delta_v^2,\ \Delta_{12} = \Delta_v^1 \cap \Delta_v^2),$$

on V_{12}, that represents each space tangent to V_{12} in the form of the direct sum of its four subsapces, hence

$$\Delta_1 \oplus \Delta_{12} = \mathrm{Ker}\,T\pi_1,\ \Delta_2 \oplus \Delta_{12} = \mathrm{Ker}\,T\pi_2,\ \Delta_1 \oplus \Delta_2 \oplus \Delta_{12} = \mathrm{Ker}\,T\sigma,$$

where $\sigma = \rho_1\pi_2 = \rho_2\pi_1$ and $\Delta \oplus \Delta_1$ and $\Delta \oplus \Delta_2$ are projected on V_1 and V_2 for $T\pi_2$ and $T\pi_1$, defining connectedness in the stratifications ρ_1 and ρ_2.

In the example under consideration, after reduction the manifold V_{12} is five-dimensional, the manifolds V_1 and V_2 are three-dimensional, and the base V is two-dimensional. The distribution Δ extended over the vector fields Y_1 and Y_2 is two-dimensional and the distributions Δ_1, Δ_2 and Δ_{12}, defined by the operators $\partial/\partial\tau_1$, $\partial/\partial\tau_2$ and $\partial/\partial\tau$, respectively, are one-dimensional. The three Pfaff equations presented above define the distribution Δ. The second and third of them define the distribution Δ_1, and the first and third, the distribution Δ_2.

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CONDITIONS FOR FINITENESS OF THE NUMBER OF INSTABILITY ZONES IN THE PROBLEM OF NORMAL VIBRATIONS OF NON-LINEAR SYSTEMS*

A.L. ZHUPIEV and YU.V. MIKHLIN

Conservative non-linear systems with two degrees of freedom that allow of normal vibrations with rectilinear trajectories in configuration space are examined. The normal vibrations of non-linear systems are a generalization for normal (principal) vibrations of linear systems /1/. The value of such solutions is determined by the fact that the resonance modes are close to normal vibrations for small external periodic effects.

A number of recent papers (/2—5/ etc.) are devoted to the analysis of normal vibrations. Within the framework of the stability problem to a first approximation, or normal vibrations, conditions are obtained for which the number of instability zones in the system parameter space is finite. The eigenfunctions and eigenvalues corresponding to the zone boundaries are determined.

1. Let the motion of a conservative system be determined by the equations

$$x_i^{\cdot\cdot} + \partial\Pi/\partial x_i = 0 \quad (i = 1, 2) \tag{1.1}$$

where $\Pi(x_1, x_2)$ is a positive-definite potential.

We assume that the system allows normal vibrations $x_2 = Cx_1$ (C is a constant). Such systems are described in /1, 3, 5/. Rotation of the coordinate axes can always result in a solution in the form $x_2 = 0$, and a system potential in the form

$$\Pi(x_1, x_2) = \sum_{i=2}^{m} a_i x_1^i + x_2^2 \sum_{i=0}^{m-2} e_i x_1^i + \sum_{i=3}^{\infty} x_2^i g_i(x_1)$$

The condition for the existence of the solutions mentioned $\partial\Pi(x_1, 0)/\partial x_2 = 0$ is satisfied identically.

Motion in time along the normal vibrations trajectory is described by a second-order equation

$$x^{\cdot\cdot} + \partial\Pi(x, 0)/\partial x = 0, \quad x \equiv x_1 \tag{1.2}$$

where the first integral (the energy integral) has the form

$$x^{\cdot 2}/2 + \Pi(x, 0) = h \tag{1.3}$$

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